1.3 Pseudodifferential operator on vector bundle

Let $P = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ be a differential operator on $\mathscr{S}$ such that $|D^\beta A^\alpha(x)|$ is bounded for all α, β. Thus from (1.2.22), for $u \in \mathscr{S}$, $Pu \in \mathscr{S}$. Then by (1.2.26) and (1.2.31), we have

$$\begin{aligned} Pu(x) = \sum_{|\alpha| \leq m} i^{|\alpha|} A^\alpha(x) D_x^\alpha u(x) &= (2\pi)^{-n/2} \sum_{|\alpha| \leq m} i^{|\alpha|} A^\alpha(x) \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \widehat{D^\alpha u}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} p(x,\xi) \hat{u}(\xi) d\xi, \quad (1.3.1) \end{aligned}$$

where

$$p(x,\xi) = \sum_{|\alpha| \leq m} i^{|\alpha|} A^\alpha(x) \xi^\alpha. \qquad (1.3.2)$$

The matrix-valued function $p(x, \xi)$ is called the (total) symbol of P. We denote by

$$\operatorname{sym}(P) = p(x, \xi). \qquad (1.3.3)$$

Let $Q : \mathscr{S} \to \mathscr{S}$ be another differential operator with symbol $q(\xi)$, i.e., A^α's are constants. Then since $\widehat{A^\alpha u}(\xi) = A^\alpha \hat{u}(\xi)$, we have $\widehat{Qu}(\xi) = q(\xi)\hat{u}(\xi)$.

$$\begin{aligned} PQu(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} p(x,\xi) \widehat{Qu}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} p(x,\xi) q(\xi) \hat{u}(\xi) d\xi. \quad (1.3.4) \end{aligned}$$

Thus

$$\operatorname{sym}(P \circ Q) = \operatorname{sym}(P) \cdot \operatorname{sym}(Q). \qquad (1.3.5)$$

In the theory of PDE, the main problem is to solve the equation

$$Pu = f. \qquad (1.3.6)$$

From (1.3.4), naively, if $p(\xi)$ is invertible and independent of x, by (1.2.24), letting $q(\xi) := p(\xi)^{-1}$ and

$$Qf(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} q(\xi) \hat{f}(\xi) d\xi, \qquad (1.3.7)$$

we have

$$PQf(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\xi = f(x). \tag{1.3.8}$$

Thus

$$u = Qf \tag{1.3.9}$$

is a solution of (1.3.6).

There are two problems for this process. Firstly, $p(\xi)$ is not invertible at $\xi = 0$. For this problem, we will assume that $p(\xi)$ is invertible outside 0, and use a cut-off function to construct the $q(\xi)$ to handle it. It is the main content in the next section. Secondly, we need to construct a home for Q living in and study the case that p, q depend on x. This is the purpose of this section.

Definition 1.3.1. Fix $m \in \mathbb{R}$. A smooth matrix-valued function $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a symbol of order m if for each α, α', there exists $C_{\alpha,\alpha'} > 0$ such that

$$|D_x^\alpha D_\xi^{\alpha'} p(x, \xi)| \le C_{\alpha,\alpha'} (1 + |\xi|)^{m-|\alpha'|} \tag{1.3.10}$$

for all x, ξ. Let Sym^m be the space of these symbols.

Now we construct the operator from such $p(x, \xi)$ as in (1.3.7).

Proposition 1.3.2. *For each $p \in \mathrm{Sym}^m$, the formula*

$$Pu(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x, \xi) \hat{u}(\xi) d\xi \tag{1.3.11}$$

defines a linear operator $P : \mathscr{S} \to \mathscr{S}$. If p has compact x-support, this operator has a continuous extension $P : \mathbf{H}^{s+m} \to \mathbf{H}^s$ for any $s \in \mathbb{R}$.

Proof. For $u \in \mathscr{S}$, $\hat{u} \in \mathscr{S}$. For $N \in \mathbb{N}$, from (1.2.22) and (1.3.10), for any $k \in \mathbb{N}$,

$$\begin{aligned}
|x|^{2N} |D_x^\alpha Pu(x)| &= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} |x|^{2N} D_x^\beta e^{i\langle x,\xi\rangle} D_x^\gamma p(x, \xi) \hat{u}(\xi) d\xi \right| \\
&= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} \left(\Delta_\xi^N e^{i\langle x,\xi\rangle}\right) (\xi^\beta D_x^\gamma p(x, \xi)) \hat{u}(\xi) d\xi \right| \\
&= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} e^{i\langle x,\xi\rangle} \Delta_\xi^N (\xi^\beta D_x^\gamma p(x, \xi) \hat{u}(\xi)) d\xi \right| \\
&\le C_k \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} (1 + |\xi|)^{m+|\beta|} (1 + |\xi|)^{-k} d\xi.
\end{aligned} \tag{1.3.12}$$

Take k large enough, the right hand side of (1.3.12) is finite. Thus $Pu \in \mathscr{S}$.

Now we prove the second part.

We firstly try to use the definition:

$$\begin{aligned}\|Pu\|_s = \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{Pu}(\xi)|^2 d\xi \\ = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1+|\xi|)^{2s} \left| \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} Pu(x)dx \right|^2 d\xi \\ = (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|)^{2s} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle} p(x,\eta)\hat{u}(\eta) d\eta dx \right|^2 d\xi \quad (1.3.13)\end{aligned}$$

But in this way, $\int_{\mathbb{R}^n}((1+|\xi|)^{2s})d\xi$ may be not finite. It is not easy for us to use $e^{-i\langle x,\xi\rangle}$ to control it.

Now we use another equivalent definition (1.2.53):

$$\|Pu\|_s = \sup_v \frac{(Pu,v)}{\|v\|_{-s}}. \quad (1.3.14)$$

From (1.2.52) and (1.3.13),

$$\begin{aligned}(Pu,v) = \int_{\mathbb{R}^n} \widehat{Pu}(\xi) \cdot \hat{v}(\xi) d\xi \\ = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle} p(x,\eta)\hat{u}(\eta) d\eta dx \right) \cdot \hat{v}(\xi) d\xi \quad (1.3.15)\end{aligned}$$

Set

$$\Psi(\xi,\eta) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle} p(x,\eta) dx. \quad (1.3.16)$$

Then

$$|(Pu,v)| \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Psi(\xi,\eta)||\hat{u}(\eta)||\hat{v}(\xi)| d\xi d\eta. \quad (1.3.17)$$

We show an estimate of $\Psi(\xi,\eta)$ as follows. For any α,

$$\begin{aligned}\zeta^\alpha \int_{\mathbb{R}^n} e^{-i\langle x,\zeta\rangle} p(x,\eta) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha e^{-i\langle x,\zeta\rangle} p(x,\eta) dx \\ = \int_{\mathbb{R}^n} e^{-i\langle x,\zeta\rangle} D_x^\alpha p(x,\eta) dx. \quad (1.3.18)\end{aligned}$$

Since p has compact x-support, by (1.3.10) and (1.3.18), for any $t \in \mathbb{N}$, there exists $C_t > 0$, such that

$$|\Psi(\xi,\eta)| \leq C_t(1+|\eta|)^m(1+|\xi-\eta|)^{-t}. \tag{1.3.19}$$

Note that

$$(1+|\eta|)(1+|\xi-\eta|) \geq 1+|\eta|+|\xi-\eta| \geq 1+|\xi|. \tag{1.3.20}$$

Thus

$$\frac{1+|\xi|}{1+|\eta|} \leq (1+|\xi-\eta|). \tag{1.3.21}$$

Let

$$\Psi'(\xi,\eta) = \Psi(\xi,\eta)\cdot(1+|\eta|)^{-s-m}\cdot(1+|\xi|)^s. \tag{1.3.22}$$

Then from (1.3.19)-(1.3.22),

$$|\Psi'(\xi,\eta)| \leq C_t\frac{(1+|\xi|)^s}{(1+|\eta|)^s}(1+|\xi-\eta|)^{-t} \leq C_t(1+|\xi-\eta|)^{|s|-t}. \tag{1.3.23}$$

Thus taking t large enough, $\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\xi$ and $\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\eta$ are all finite.

From (1.3.17) and (1.3.22), we have

$$\begin{aligned}|(Pu,v)| \leq & \left(\int_{R^n}\left(\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\xi\right)(1+|\eta|)^{s+m}\hat{u}(\eta)d\eta\right)^{1/2}\\ & \times\left(\int_{R^n}\left(\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\eta\right)(1+|\xi|)^{-s}\hat{v}(\xi)d\xi\right)^{1/2}\\ & \leq C\|u\|_{s+m}\|v\|_{-s}.\end{aligned} \tag{1.3.24}$$

Therefore, we get this proposition from (1.3.14). □

Remark that the constant C in (1.3.24) depends on s.

Definition 1.3.3. The operator P defined in (1.3.11) is called a **pseudodifferential operator of order** m on $\mathbb{R}^n$. In particular, a differential operator is a pseudodifferential operator. The space of the pseudodifferential operators of order m is denoted by $\Psi\mathrm{DO}_m$. A linear map $f : \mathscr{S} \to \mathscr{S}$ is called an **(infinitely) smoothing operator** if it could extend to $f : \mathbf{H}^s \to \mathbf{H}^{s+m}$ for any s and m. Two pseudodifferential operators P and P' are called equivalent if $P - P'$ is an (infinitely) smoothing operator.

Our next aim is to define the composition of $P, P' \in \Psi\mathrm{DO}$. This is one of the key point in solving (1.3.6).

Since the pseudodifferential operator is defined by the symbol, we study the "symbol calculus".

Definition 1.3.4. Let P be a pseudodifferential operator with symbol p. Then p is said to have a formal development

$$p \sim \sum_{j=1}^{\infty} p_j, \quad p_j \in \mathrm{Sym}^{m_j}, \tag{1.3.25}$$

if for each $m \in \mathbb{Z}$, there exists K such that $p - \sum_{j=1}^{k} p_j \in \mathrm{Sym}^{-m}$ for any $k \geq K$.

The following proposition says that the set of symbols is complete under the addition in some sense.

Proposition 1.3.5. *Any formal series* $\sum_{j=1}^{\infty} p_j$, $p_j \in \mathrm{Sym}^{m_j}$, $m_j \to -\infty$, *is the formal development of a pseudodifferential operator. This operator is unique up to equivalence.*

Proof. We can assume that $m_{j+1} < m_j$ for all j. Fix a smooth function $\varphi : \mathbb{R}^n \to [0,1]$, such that $\varphi(\xi) = 0$ for $|\xi| \leq 1$ and $\varphi(x) = 1$ for $|\xi| \geq 2$. For any sequence $\{r_j\}_{j=1}^{\infty}$ such that $\lim_{t\to\infty} r_j = +\infty$, the symbol

$$p(x,\xi) = \sum_{j=1}^{\infty} \varphi(\xi/r_j) p_j(x,\xi) \tag{1.3.26}$$

is well defined since the sum is finite for each (x,ξ). We plan to choose r_j such that $p(x,\xi)$ is the symbol of a pseudodifferential operator.

From (1.3.10), we have

$$\begin{aligned} |D_x^{\alpha} D_\xi^{\alpha'} (\varphi(\xi/r_j) p_j(x,\xi))| &\leq C_{\alpha'} \sum_{\beta+\gamma=\alpha'} \left| D_\xi^{\beta} \varphi(\xi/r_j) \right| \cdot \left| D_x^{\alpha} D_\xi^{\gamma} p_j(x,\xi) \right| \\ &\leq C_{\alpha'} \sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma} \left| D_\xi^{\beta} \varphi(\xi/r_j) \right| \cdot (1+|\xi|)^{m_j-|\gamma|}. \end{aligned} \tag{1.3.27}$$

Let $\tilde{\varphi}_j(\xi) = \varphi(\xi/r_j)$. By induction, we could prove that

$$\left| D_\xi^{\beta} \varphi(\xi/r_j) \right| \leq \frac{C_\beta}{r_j^{|\beta|}} \|\tilde{\varphi}_j\|_{\mathscr{C}^{|\beta|}} \leq \frac{C_\beta}{r_j^{|\beta|}} \|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}}. \tag{1.3.28}$$

Thus

$$|D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| \le C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \sum_{\beta+\gamma=\alpha'} \frac{C_{j,\alpha,\gamma}C_\beta}{r_j^{|\beta|}} \cdot (1+|\xi|)^{m_j-|\gamma|}. \tag{1.3.29}$$

Note that if $|\xi| < r_j$, $D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big) = 0$. Thus we could assume $|\xi| \ge r_j$ in the right hand side of (1.3.29). Set $m_0 = m_1$. Then

$$\begin{aligned} |D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| &\le C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \sum_{\beta+\gamma=\alpha'} \frac{C_{j,\alpha,\gamma}C_\beta}{r_j^{|\beta|}} \cdot \frac{(1+|\xi|)^{m_{j-1}-|\alpha'|}}{(1+|\xi|)^{m_{j-1}-m_j-|\beta|}} \\ &\le C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \frac{\sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta}{r_j^{m_{j-1}-m_j}} \cdot (1+|\xi|)^{m_{j-1}-|\alpha'|}. \end{aligned} \tag{1.3.30}$$

Let $C_{j,\alpha,\alpha'} := \sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta$. For $j > 1$, set $r_j > (2^j\|\tilde{\varphi}_j\|_{\mathscr{C}^j} \max_{|\alpha|,|\alpha'|\le j}\{C_{j,\alpha,\alpha'}\})^{1/(m_{j-1}-m_j)}$ such that $\lim_{j\to+\infty} r_j = +\infty$. Then if $|\alpha|, |\alpha'| \le j$, there exists $C_{\alpha,\alpha'} > 0$ such that

$$|D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| \le \frac{C_{\alpha,\alpha'}}{2^j}(1+|\xi|)^{m_{j-1}-|\alpha'|}. \tag{1.3.31}$$

Let $k = \max\{|\alpha|, |\alpha'|\}$. Then there exists $C_{\alpha,\alpha'} > 0$ such that

$$\begin{aligned} |D_x^\alpha D_\xi^{\alpha'} p(x,\xi))| &\le \left(\sum_{j=1}^{k} C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \frac{\sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta}{r_j^{m_{j-1}-m_j}}\right)(1+|\xi|)^{m_1-|\alpha'|} \\ &\quad + \sum_{j=k+1}^{\infty} \frac{C_{\alpha,\alpha'}}{2^j}(1+|\xi|)^{m_1-|\alpha'|} \le C_{\alpha,\alpha'}(1+|\xi|)^{m_1-|\alpha'|}. \end{aligned} \tag{1.3.32}$$

Therefore, $p \in \mathrm{Sym}^{m_1}$.

Following the same process, we could obtain that $p - \sum_{j=1}^k \tilde{\varphi}_j p_j \in \mathrm{Sym}^{m_k}$. Since $m_j \to -\infty$, for any $m \in \mathbb{Z}$, there exists $K > 0$ such that for any $k > K$, $p - \sum_{j=1}^k \tilde{\varphi}_j p_j \in \mathrm{Sym}^m$. Since $(1-\tilde{\varphi}_j)p_j \in \mathrm{Sym}^{-\infty}$, we have $p - \sum_{j=1}^k p_j \in \mathrm{Sym}^m$. Thus $\sum_{j=1}^\infty p_j$ is the formal development of a pseudodifferential operator.

If P' is another pseudodifferential operator with symbol p' has the same formal development, then for any $m \in \mathbb{Z}$, $p - p' \in \mathrm{Sym}^{-m}$. Thus $p - p' \in \mathrm{Sym}^{-\infty}$.

The proof of Proposition 1.3.5 is completed. □

From (1.3.1), we have

$$(Pu)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}p(x,\xi)u(y)dyd\xi. \tag{1.3.33}$$

The following lemma is technical. But in Shubin's famous book, he strongly urges the reader to carefully look at this proof. Therefore we present it here.

Lemma 1.3.6. *Let $a(x,y,\xi)$ be a smooth matrix-valued function on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with compact x- and y-support. Fix $m \in \mathbb{R}$ and assume that for each α,β,γ, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a| \leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{m-|\gamma|}. \tag{1.3.34}$$

Then the operator $K : \mathscr{S} \to \mathscr{S}$ given by

$$(Ku)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}a(x,y,\xi)u(y)dyd\xi \tag{1.3.35}$$

is a pseudodifferential operator whose symbol k has asymptotic development

$$k(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha a)(x,x,\xi). \tag{1.3.36}$$

Proof. From (1.2.24), (1.2.29), (1.2.30) and (1.3.35), we have

$$\begin{aligned}(Ku)(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\left(\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}a(x,y,\xi)u(y)dy\right)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\widehat{a_y u}(\xi)d\xi = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}(\hat{a}_y * \hat{u})(\xi)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\int_{\mathbb{R}^n}\hat{a}_y(x,\xi-\eta,\xi)\hat{u}(\eta)d\eta d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle}\left(\int_{\mathbb{R}^n} e^{i\langle x,\xi-\eta\rangle}\hat{a}_y(x,\xi-\eta,\xi)d\xi\right)\hat{u}(\eta)d\eta. \end{aligned} \tag{1.3.37}$$

We need to check the interchange of integrations in the last equation is allowed. From (1.3.16) and (1.3.19), since a is with compact x- and y-support, for any $l_1 \in \mathbb{N}$, there exists $C_{l_1} > 0$ such that

$$\begin{aligned}|\hat{a}_y(x,\xi-\eta,\xi)| &= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{i\langle \eta-\xi,s\rangle}a(x,s,\xi)ds\right| \\ &\leq C_{l_1}(1+|\xi|)^m(1+|\xi-\eta|)^{-l_1}. \end{aligned} \tag{1.3.38}$$

Since $\hat{u} \in \mathscr{S}$, for any $l_2 \in \mathbb{N}$, there exists $C_{l_2} > 0$ such that

$$|\hat{u}(\eta)| \leq C_{l_2}(1+|\eta|)^{-l_2}. \tag{1.3.39}$$

Thus from (1.3.21), for $l_1, l_2 \in \mathbb{N}$, such that $m+n/2 < l_1 < l_2 - n/2$, we have

$$|\hat{a}_y(x, \xi-\eta, \xi)||\hat{u}(\eta)| \leq C_{l_1}C_{l_2}(1+|\xi|)^{m-l_1}(1+|\eta|)^{l_1-l_2}. \tag{1.3.40}$$

Thus the interchange of integrations is allowed.

Let

$$k(x,\eta) = \int_{\mathbb{R}^n} e^{i\langle x, \xi-\eta\rangle}\hat{a}_y(x,\xi-\eta,\xi)d\xi = \int_{\mathbb{R}^n} e^{i\langle x,\zeta\rangle}\hat{a}_y(x,\zeta,\zeta+\eta)d\zeta. \tag{1.3.41}$$

Then from (1.3.11), if $k(x,\eta)$ satisfies (1.3.10), K is a pseudodifferential operator with symbol $k(x,\eta)$.

For $l \in \mathbb{N}$, we have the Taylor expansion in the third variable,

$$\hat{a}_y(x,\zeta,\zeta+\eta) = \sum_{|\alpha|\leq l} \frac{i^{|\alpha|}}{\alpha!}(D^\alpha_\eta \hat{a}_y)(x,\zeta,\eta)\zeta^\alpha + R_l(x,\zeta,\zeta+\eta). \tag{1.3.42}$$

***Remark of Taylor expansion**: For function $F \in \mathscr{S}$, the Taylor expansion is*

$$F(x) = \sum_{|\alpha|\leq l} \frac{1}{\alpha!}F^{(\alpha)}(0)x^\alpha + R_l(x), \tag{1.3.43}$$

where

$$R_l(x) = \sum_{|\mu|=l+1} \frac{l+1}{\mu!}x^\mu \cdot \int_0^1 (1-t)^l F^{(\mu)}(tx)dt. \tag{1.3.44}$$

We explain (1.3.44) here in some details. By Taylor extension and the integration remainder in one variable, for $\varphi \in \mathscr{C}^\infty(\mathbb{R})$, we have

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + r_k(t),$$

where

$$r_k(t) = \frac{1}{k!}\int_0^t (t-s)^k \varphi^{(k+1)}(s)ds.$$

Set $\varphi(t) = F(tx)$. Then

$$\varphi'(t) = \sum_{|J|=1} F^{(J)}(tx)x^J, \quad \varphi''(t) = \sum_{|J|=1}\sum_{|K|=1} F^{(J+K)}(tx)x^{J+K} = \sum_{|J|=2} F^{(J)}(tx)x^J.$$

By induction, we have

$$\varphi^{(k)}(t) = \sum_{|J|=k} F^{(J)}(tx)x^J.$$

So we have

$$F(x)=\sum_{|J|\le l}\frac{1}{|J|!}F^{(J)}(0)x^J+\frac{1}{k!}\int_0^1(1-s)^k\sum_{|J|=k+1}F^{(J)}(sx)x^J ds.$$

Note that

$$\sum_{|J|=k}F^{(J)}x^J=\sum_{|\alpha|=k}\nu(\alpha)F^{(\alpha)}x^\alpha,$$

where

$$\nu(\alpha)=\binom{k}{\alpha_1}\cdot\binom{k-\alpha_1}{\alpha_2}\cdots\binom{k-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n}=\frac{k!}{\alpha!}.$$

Thus we get (1.3.43) and (1.3.44).

By (1.3.44),

$$R_l(x,\zeta,\zeta+\eta)=\sum_{|\mu|=l+1}\frac{(l+1)i^{l+1}}{\mu!}\int_0^1(1-t)^l(D^\mu_\eta\hat{a}_y)(x,\zeta,t\zeta+\eta)\zeta^\mu dt. \tag{1.3.45}$$

Since

$$\begin{aligned}\int_{\mathbb{R}^n}e^{i\langle x,\zeta\rangle}(D^\alpha_\eta\hat{a}_y)(x,\zeta,\eta)\zeta^\alpha d\zeta&=\int_{\mathbb{R}^n}e^{i\langle x,\zeta\rangle}\zeta^\alpha(\widehat{D^\alpha_\eta a_y})(x,\zeta,\eta)d\zeta\\&=\int_{\mathbb{R}^n}e^{i\langle x,\zeta\rangle}(\widehat{D^\alpha_y D^\alpha_\eta a_y})(x,\zeta,\eta)d\zeta=(D^\alpha_y D^\alpha_\eta a_y)(x,x,\eta),\end{aligned} \tag{1.3.46}$$

from (1.3.41), (1.3.42), (1.3.45) and (1.3.46), we have

$$k(x,\eta)=\sum_{|\alpha|\le l}\frac{i^{|\alpha|}}{\alpha!}(D^\alpha_y D^\alpha_\eta a_y)(x,x,\eta)+r_l(x,\eta), \tag{1.3.47}$$

where

$$r_l(x,\eta)=\int_{\mathbb{R}^n}e^{i\langle x,\zeta\rangle}R_l(x,\zeta,\zeta+\eta)d\zeta. \tag{1.3.48}$$

By Proposition 1.3.5, we only need to prove that for any $l\in\mathbb{N}$, $r_l(x,\eta)\in\mathrm{Sym}^{m-(l+1)}$.

From (1.3.45) and (1.3.48),

$$\begin{aligned}|D^\alpha_x D^\beta_\eta r_l(x,\eta)|&\le\int_{\mathbb{R}^n}|D^\alpha_x D^\beta_\eta R_l(x,\zeta,\zeta+\eta)|d\zeta\\&\le\sum_{|\mu|=l+1}\frac{(l+1)}{\mu!}\int_{\mathbb{R}^n}\int_0^1(1-t)^l|(D^\alpha_x D^{\mu+\beta}_\eta\hat{a}_y)(x,\zeta,t\zeta+\eta)\zeta^\mu|d\zeta dt.\end{aligned} \tag{1.3.49}$$

From (1.3.34),

$$\begin{aligned}
|\zeta^\gamma(D_x^\alpha D_\eta^\beta \hat{a}_y)(x,\zeta,t\zeta{+}\eta)| &= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{-i\langle y,\zeta\rangle}\zeta^\gamma(D_x^\alpha D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} D_y^\gamma e^{-i\langle y,\zeta\rangle}(D_x^\alpha D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{-i\langle y,\zeta\rangle}(D_x^\alpha D_y^\gamma D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&\le C_{\alpha\beta\gamma}\operatorname{vol}(y-\operatorname{supp}(a))(1+|t\zeta+\eta|)^{m-|\beta|} \\
&\le C_{\alpha\beta\gamma}(1+|t\zeta+\eta|)^{m-|\beta|}. \quad (1.3.50)
\end{aligned}$$

From (1.3.49) and (1.3.50), taking $l+1>m$, $\gamma>n+2(l+1)+|\beta|+m$,

$$\begin{aligned}
|D_x^\alpha D_\eta^\beta r_l(x,\eta)| &\le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}\int_0^1 (1{-}t)^l(1{+}|t\zeta{+}\eta|)^{m-(l+1)-|\beta|}(1{+}|\zeta|)^{-\gamma}|\zeta|^{l+1}d\zeta dt \\
&\le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}\int_0^1 (1-t)^l(1+|\eta|)^{m-(l+1)-|\beta|}(1+t|\zeta|)^{l+1+|\beta|-m}(1+|\zeta|)^{l+1-\gamma}d\zeta dt \\
&\le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}(1+|\eta|)^{m-(l+1)-|\beta|}(1+|\zeta|)^{2(l+1)+|\beta|-m-\gamma}d\zeta \\
&\le C_{\alpha\beta\gamma l}(1+|\eta|)^{m-(l+1)-|\beta|}. \quad (1.3.51)
\end{aligned}$$

The proof of Lemma 1.3.6 is completed. □

Remark 1.3.7. If the function $a(x,y,\xi)$ in Lemma 1.3.6 vanishes for all (x,y) in a neighborhood of the diagonal, then the corresponding operator K given by (1.3.35) is infinitely smoothing.

For a differential operator P, from the definition, we see easily that P is local, that is, for $u\in\mathscr{C}_0^\infty$,

$$\operatorname{supp}(Pu)\subset\operatorname{supp}(u). \tag{1.3.52}$$

Remark that a surprising theorem by Peetre says that if a linear operator satisfies (1.3.52), it is a differential operator. So we cannot expect the pseudodifferential operator is local. But we can prove that it is ε-local up to a smoothing operator.

For $A\subset\mathbb{R}^n$ and $\varepsilon>0$, we set

$$A_\varepsilon := \{x\in\mathbb{R}^n : \operatorname{dist}(x,A)\le\varepsilon\}. \tag{1.3.53}$$

An operator $P:\mathscr{S}\to\mathscr{S}$ is called ε-local if for any $u\in\mathscr{C}_0^\infty$,

$$\operatorname{supp}(Pu)\subset\operatorname{supp}(u)_\varepsilon. \tag{1.3.54}$$

Proposition 1.3.8. *For $P \in \Psi\mathrm{DO}_m$ with symbol p, which has compact x-support, for any $\varepsilon > 0$, there exists $P_\varepsilon \in \Psi\mathrm{DO}_m$ such that it is equivalent to P and ε-local.*

Proof. Choose a smooth real-valued function ψ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\psi \equiv 1$ in a neighborhood of the diagonal and $\psi(x, y) = 0$ if $|x - y| \geq \varepsilon$. Then

$$(P_\varepsilon u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \psi(x, y)p(x, \xi)u(y)dyd\xi \tag{1.3.55}$$

is ε-local. By (1.3.33) and Lemma 1.3.6, P_ε is a pseudodifferential operator with symbol $p_\varepsilon \sim p$. Thus P_ε is equivalent to P.

The proof of Proposition 1.3.8 is completed. □

Proposition 1.3.9. *Let $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$. For $P \in \Psi\mathrm{DO}_m$,*

$$P^{\chi_1,\chi_2}(u) := \chi_1 P(\chi_2 u) \in \Psi\mathrm{DO}_m\,. \tag{1.3.56}$$

Proof. From (1.3.56),

$$(P^{\chi_1,\chi_2}u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \chi_1(x)p(x, \xi)\chi_2(y)u(y)dyd\xi. \tag{1.3.57}$$

Set $a(x, y, \xi) = \chi_1(x)p(x, \xi)\chi_2(y)$ in Lemma 1.3.6, we obtain Proposition 1.3.9. □

Proposition 1.3.10. *Let P be a pseudodifferential operator and $u \in \mathbf{H}^s$ for some $s \in \mathbb{R}$. Then for any open subset $U \subset \mathbb{R}^n$, if $u|_U \in \mathscr{C}^\infty$, then $Pu|_U \in \mathscr{C}^\infty$.*

Proof. For $x \in U$, choose $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that $x \in \mathrm{supp}(\chi_1) \subset \mathrm{supp}(\chi_2)$, $\chi_1 \equiv 1$ near x and $\chi_2 \equiv 1$ on a neighborhood of $\mathrm{supp}(\chi_1)$. By Proposition 1.3.2, we have $P(\chi_2 u) \in \mathscr{C}^\infty$. Thus $\chi_1 P(\chi_2 u) \in \mathscr{C}^\infty$. Since

$$\chi_1 P((1 - \chi_2)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \chi_1(x)p(x, \xi)(1 - \chi_2)(y)u(y)dyd\xi, \tag{1.3.58}$$

and $\chi_1(x)(1-\chi_2)(y)$ vanishes in a neighborhood of the diagonal, by Remark 1.3.7, we have $\chi_1 P((1 - \chi_2)u) \in \mathscr{C}^\infty$. Therefore, $\chi_1 Pu \in \mathscr{C}^\infty$, which means that Pu is smooth near x. □

Note that the pseudodifferential operator is not local. Sometimes, in order to define it on manifolds, we need some conditions to guarantee that we can do analysis on a local chart.

Definition 1.3.11. For $P \in \Psi\mathrm{DO}_m$, if there exists a compact subset $K \subset \mathbb{R}^n$ such that for any $u \in \mathscr{C}_0^\infty$, $\mathrm{supp}(Pu) \subset K$ and $Pu = 0$ whenever $\mathrm{supp}(u) \cap K = \emptyset$, we say P has support in K. The set of such operators is denoted by $\Psi\mathrm{DO}_{K,m}$.

If P is an m-th order differential operator with the supports of all coefficients in K, then $P \in \Psi\mathrm{DO}_{K,m}$.

Let $K' \subset K$ be a compact subset such that there exists $\varepsilon > 0$ such that $\mathrm{dist}(K', \partial K) > \varepsilon$. Let $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ such that $\psi(x,y) = 0$ if $x \notin K'$ or $|x-y| \geq \varepsilon$. Then for any $p \in \mathrm{Sym}^m$, the pseudodifferential operator associated with $\psi(x,y)p(x,\xi)$ in (1.3.35) is an element of $\Psi\mathrm{DO}_{K,m}$.

Definition 1.3.12. For $P \in \Psi\mathrm{DO}_{K,m}$, its formal adjoint P^* is defined by

$$(Pu, v)_{L^2} = (u, P^*v)_{L^2}, \tag{1.3.59}$$

for any $u, v \in \mathscr{C}_0^\infty(K)$.

Proposition 1.3.13. *For $P \in \Psi\mathrm{DO}_{K,m}$ with symbol p, its formal adjoint $P^* \in \Psi\mathrm{DO}_{K,m}$ has symbol p^* with formal development*

$$p^* \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \bar{p}^T, \tag{1.3.60}$$

where $(\cdot)^T$ denotes the transposed matrix. In particular, the formal adjoint is unique up to smoothing operators.

Proof. For $u, v \in \mathscr{C}_0^\infty(K)$,

$$\begin{aligned}(Pu, v)_{L^2} &= \int_{\mathbb{R}^n} \left\langle (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)u(y)dyd\xi, v(x) \right\rangle dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \langle p(x,\xi)u(y), v(x)\rangle dyd\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle u(y), e^{-i\langle x-y,\xi\rangle} \overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx. \end{aligned} \tag{1.3.61}$$

Formally, we could write $P^*v = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x-y,\xi\rangle}\overline{p(x,\xi)}^T v(x) d\xi dx$. But usually, $\overline{p(x,\xi)}^T$ does not satisfy (1.3.34)the condition of Lemma 1.3.6. We use the following trick to overcome this obstruction. Fix $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\phi \equiv 1$ on K. Then $\phi u = u$. From (1.3.61), we have

$$\begin{aligned}(Pu, v)_{L^2} &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle \phi(y)u(y), e^{-i\langle x-y,\xi\rangle} \overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle u(y), e^{-i\langle x-y,\xi\rangle} \phi(y)\overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx. \end{aligned} \tag{1.3.62}$$

In this case, from (1.3.10), we have

$$|D_x^\alpha D_y^\beta D_\xi^\gamma \phi(y)\overline{p(x,\xi)}^T| \leq C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\gamma|}. \tag{1.3.63}$$

Thus from Lemma 1.3.6,

$$(P^*v)(y) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle}\phi(y)\overline{p(x,\xi)}^T v(x)d\xi dx, \tag{1.3.64}$$

satisfying $(Pu,v)_{L^2} = (u,P^*v)_{L^2}$, is a pesudodifferential operator with symbol

$$p^*(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha \phi(x)\overline{p(y,\xi)}^T\bigg|_{x=y} = \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}^T. \tag{1.3.65}$$

The last equality holds because $p(x,\xi)$ has x-support in K.

If P_1^* is another formal adjoint of P, then the symbol of it has the same formal development as p^*. Thus $P_1^* - P^*$ is a smoothing operator.

The proof of Proposition 1.3.13 is completed. □

From Proposition 1.3.13, we could obtain another formula of $\widehat{Pu}(\xi)$ as follows. For $u, v \in \mathscr{S}$,

$$\begin{aligned}(\widehat{Pu},\hat{v})_{L^2} = (u,P^*v)_{L^2} &= \int_{\mathbb{R}^n}\left\langle u(x),(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}p^*(x,\xi)\hat{v}(\xi)d\xi\right\rangle dx \\ &= \int_{\mathbb{R}^n}\left\langle (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\overline{p^*(x,\xi)}^T u(x)dx, \hat{v}(\xi)\right\rangle d\xi. \end{aligned}\tag{1.3.66}$$

Therefore, we have

$$\widehat{Pu}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\overline{p^*(x,\xi)}^T u(x)dx. \tag{1.3.67}$$

Proposition 1.3.14. *For $P \in \Psi\mathrm{DO}_{K,l}$ and $Q \in \Psi\mathrm{DO}_{K,m}$ with symbols p and q respectively, the composition $P \circ Q \in \Psi\mathrm{DO}_{K,l+m}$ has symbol*

$$\mathrm{Sym}(P\circ Q) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p)(D_x^\alpha q). \tag{1.3.68}$$

Proof. By (1.3.11) and (1.3.67), we have

$$\begin{aligned}(PQu)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}p(x,\xi)\widehat{Qu}(\xi)d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}p(x,\xi)\overline{q^*(y,\xi)}^T u(y)dyd\xi. \end{aligned}\tag{1.3.69}$$

Since $P \in \Psi\mathrm{DO}_{K,l}$ and $Q \in \Psi\mathrm{DO}_{K,m}$, there exists $C_{\alpha\beta\gamma} > 0$, such that

$$|D_x^\alpha D_y^\beta D_\xi^\gamma p(x,\xi)\overline{q^*(y,\xi)}^T| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{m+l-|\gamma|}. \tag{1.3.70}$$

Thus by Lemma 1.3.6, $P \circ Q \in \Psi\mathrm{DO}_{K,l+m}$.

Since $(P^*)^* = P$, by (1.3.36) and (1.3.60), we have

$$\begin{aligned}
\mathrm{Sym}(PQ) &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha p(x,\xi)\overline{q^*(y,\xi)}^T)|_{x=y} \\
&= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(D_\xi^\beta p(x,\xi))(D_\xi^\gamma D_x^\alpha \overline{q^*(x,\xi)}^T) \\
&= \sum_\alpha \sum_{\beta+\gamma=\alpha} \frac{i^{|\beta|+|\gamma|}}{\beta!\gamma!}(D_\xi^\beta p(x,\xi))(D_\xi^\gamma D_x^\beta D_x^\gamma \overline{q^*(x,\xi)}^T) \\
&= \sum_\beta \frac{i^{|\beta|}}{\beta!}(D_\xi^\beta p(x,\xi)) D_x^\beta \left(\sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma D_x^\gamma \overline{q^*(x,\xi)}^T \right) \\
&\sim \sum_\beta \frac{i^{|\beta|}}{\beta!}(D_\xi^\beta p(x,\xi))(D_x^\beta q(x,\xi)).
\end{aligned} \tag{1.3.71}$$

The proof of Proposition 1.3.14 is completed. □

Proposition 1.3.15. *Let $\phi : U \to V$ be a diffeomorphism between two open sets of $\mathbb{R}^n$. Then for each compact subset $K \subset U$, ϕ induces a map $\phi_* : \Psi\mathrm{DO}_{K,m} \to \Psi\mathrm{DO}_{\phi(K),m}$ by*

$$(\phi_* P)(u) := P(u \circ \phi) \circ \phi^{-1}. \tag{1.3.72}$$

Proof. Let $\psi = \phi^{-1}$. For $x \in \phi(K)$, write $x' = \psi(x)$. Then

$$\begin{aligned}
x' - y' = \psi(x) - \psi(y) &= \int_0^1 \frac{d}{dt}\psi(tx+(1-t)y)dt \\
&= \int_0^1 \nabla\psi(tx+(1-t)y)dt \cdot (x-y).
\end{aligned} \tag{1.3.73}$$

Set $\Psi(x,y) := \int_0^1 \nabla\psi(tx+(1-t)y)dt$. Then it is a smooth matrix-valued function. Since $\Psi(x,x) = (\partial\psi^i/\partial x^j)_x$ and ψ is a diffeomorphism, the matrix $\Psi(x,y)$ is invertible in a neighborhood U of the diagonal.

Let $J(x) := |\det(\partial\psi^i/\partial x^j)_x|$ denote the Jacobian of ψ. Let $\chi \in \mathscr{C}_0^\infty(U)$ such that $\chi \equiv 1$ on a smaller neighborhood of the diagonal. Then for $P \in$

$\Psi\mathrm{DO}_{K,m}$ and $u \in \mathscr{C}_0^\infty(\phi(K), \mathbb{C}^p)$,

$$\begin{aligned}
[(\phi_* P)u](x) &= [P(u \circ \phi)](x') \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x'-y',\xi\rangle}(\phi^*\chi + \phi^*(1-\chi))(x',y')p(x',\xi)u(\phi(y'))dy'd\xi \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\Psi(x,y)^T\xi\rangle}\chi(x,y)p(\psi(x),\xi)u(y)J(y)dyd\xi \\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x'-y',\xi\rangle}\phi^*(1-\chi)(x',y')p(x',\xi)u(\phi(y'))dy'd\xi \\
&\qquad\qquad =: (P_1 u)(x) + (P_2 u)(x). \qquad (1.3.74)
\end{aligned}$$

By Remark 1.3.7, since P has compact support, P_2 is a smoothing operator. Let $\zeta = \Psi(x,y)^T\xi$. Then

$$\begin{aligned}
P_1 u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} & e^{i\langle x-y,\zeta\rangle}\chi(x,y)p(\psi(x),(\Psi(x,y)^T)^{-1}\zeta)J(y) \\
& \cdot |\det(\Psi(x,y)^T)^{-1}|u(y)dyd\zeta. \qquad (1.3.75)
\end{aligned}$$

Write

$$a(x,y,\zeta) := \chi(x,y)p(\psi(x),(\Psi(x,y)^T)^{-1}\zeta)J(y)|\det(\Psi(x,y)^T)^{-1}|. \qquad (1.3.76)$$

Then by Lemma 1.3.6, we see that $P_1 \in \Psi\mathrm{DO}_m$. It is easy to verify that P_1, P_2 have compact supports in $\phi(K)$.

The proof of Proposition 1.3.15 is completed. □

Now we define the pseudodifferential operator on manifolds.

Let M be a smooth manifold. Let E and F be complex vector bundles over M.

Definition 1.3.16. A continuous[1] linear map $P : \mathscr{C}_0^\infty(M,E) \to \mathscr{C}^\infty(M,F)$ is called a pseudodifferential operator of order m if for each coordinate system $\{U_i, \phi_i\}$, for any $\varphi, \psi \in \mathscr{C}_0^\infty(U_i)$, $(\phi_i^{-1})^*(\varphi P \psi)$ is a pseudodifferential operator of order m. The linear space of all such operators is denoted by $\Psi\mathrm{DO}_m(E,F)$. The element in $\Psi\mathrm{DO}_{-\infty}(E,F)$ is called a smoothing operator. Two pseudodifferential operators are called equivalent if they differ by a smoothing operator.

---

[1] $\{\varphi_k\} \subset \mathscr{C}_0^\infty$ tends to 0 if there exists a compact subset $K \subset \mathbb{R}^n$ such that $\mathrm{supp}(\varphi_k) \subset K$ for any k and for any α, $D^\alpha\varphi_k \to 0$ uniformly on $x \in K$. With this topology, $\mathscr{C}_0^\infty$ is usually denoted by $\mathscr{D}$. $\{\varphi_k\} \subset \mathscr{C}^\infty$ tends to 0 if for any compact subset $K \subset \mathbb{R}^n$, for any $\varepsilon > 0$ and α, there exists $N > 0$ such that if $k \geq N$, $\sup_K |D^\alpha\varphi_k| < \varepsilon$. With this topology, $\mathscr{C}^\infty$ is usually denoted by $\mathscr{E}$. We assume that $P : \mathscr{D} \to \mathscr{E}$ is continuous.

By this definition, a differential operator is a pseudodifferential operator.

Proposition 1.3.17. *Let* $P \in \Psi\mathrm{DO}_m(E,F)$. *For any open set* $U \subset M$, $u|_U \in \mathscr{C}^\infty$ *implies* $Pu|_U \in \mathscr{C}^\infty$.

Proof. Let $\{U_i, \phi_i\}$ be a locally finite local coordinate system. Take $x \in U$, $x \in U_{i_0}$. We take a refinement $\{V_j\}$ of $\{U_i\}$, such that $x \in V_{j_0} \subset U_{i_0}$, $x \notin \overline{V_j}$ for any $j \neq j_0$ and if $V_k \cap V_{j_0} \neq \emptyset$, $V_k \subset U_{i_0}$. Let $\{h_j\}$ be a partition of unity with respect to $\{V_j\}$. if u is smooth near x, then so is $h_{j_0}u$. By Proposition 1.3.10, for any j, k, $h_j P h_k(h_{j_0}u)$ is smooth near x. In fact, if $V_j \cap V_{j_0} \neq \emptyset$ and $V_k \cap V_{j_0} \neq \emptyset$, $h_j, h_k \in \mathscr{C}_0^\infty(U_i)$, thus $h_j P h_k \in \Psi\mathrm{DO}_m$. For other cases, $h_j P h_k(h_{j_0}u) = 0$ near x.

Therefore, $Pu = \sum_{i,j,k} h_j P h_k(h_{j_0}u)$ is smooth near x.

The proof of Proposition 1.3.17 is completed. □

Proposition 1.3.18. *(1) If for any* $s, m \in \mathbb{R}$, *any compact set* $K \subset M$, P *extends to a bounded linear map* $P : \mathbf{H}_0^s(K,E) \to \mathbf{H}^{s-m}(K,F)$, *then* P *is a smoothing operator.*

(2) Let $\varphi, \psi \in \mathscr{C}_0^\infty(M)$. *If* $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) = \emptyset$, *then* $\varphi P \psi$ *is a smoothing operator for any* $P \in \Psi\mathrm{DO}(E,F)$.

Proof. (1) For any $u \in \mathbf{H}_0^s(K,E)$, $Pu \in \mathscr{C}^\infty(K,F)$. Thus for any $\varphi, \psi \in \mathscr{C}_0^\infty(U_i)$, $\varphi P \psi u \in \mathscr{C}^\infty(U_i, F)$. Thus $\varphi P \psi$ is a smoothing operator. By Definition 1.3.16, P is a smoothing operator.

(2) For any $u \in \mathbf{H}_0^s(K,E)$, for any $x \in \mathrm{supp}(\varphi)$, $\psi u = 0$ near x. By Proposition 1.3.17, $\varphi P \psi u$ is smooth near x. If $x \notin \mathrm{supp}(\varphi)$, $\varphi P \psi u(x) = 0$. Thus $\varphi P \psi u$ is smooth on M. So $\varphi P \psi$ is a smoothing operator. □

Given a Riemannian volume element dv on M, we can define a formal adjoint $P^* : \mathscr{C}_0^\infty(M,F^*) \to \mathscr{C}^\infty(M,E^*)$ of operator $P : \mathscr{C}_0^\infty(M,E) \to \mathscr{C}^\infty(M,F)$ by

$$\int_M \langle Pu, v\rangle dv = \int_M \langle u, P^* v\rangle dv \tag{1.3.77}$$

for $u \in \mathscr{C}_0^\infty(M,E)$ and $v \in \mathscr{C}_0^\infty(M,F^*)$.

Theorem 1.3.19. *Let* E, F *and* G *be vector bundles over a smooth manifold* M *and let* $P \in \Psi\mathrm{DO}_m(E,F)$ *and* $Q \in \Psi\mathrm{DO}_l(F,G)$ *with the compact support* K. *The following statements hold:*

(1) The operator P *extends to a bounded linear map* $P : \mathbf{H}_0^s(K,E) \to \mathbf{H}^{s-m}(K,F)$ *for any* $s \in \mathbb{R}$.

(2) $Q \circ P \in \Psi\mathrm{DO}_{m+l}(E,G)$.

(3) $P^* \in \Psi\mathrm{DO}_m(F^*, E^*)$ *for any dv.*

(4) A diffeomorphism $\phi : M \to M$ *induces a linear map* $\phi_* : \Psi\mathrm{DO}_m(\phi^* E, \phi^* F) \to \Psi\mathrm{DO}_m(E, F)$ *by* $\phi^*[(\phi_* P)u] = P(\phi^* u)$*.*

Proof. (1) Let $\{U_i, \phi_i\}$ be a local coordinate system. We only consider the part which covers K. We could assume that $U_i = \phi_i^{-1}(B_0(4))$, $V_i = \phi_i^{-1}(B_0(1))$ and $K \subset \cup V_i$. Let $\{h_i\}$ be a partition of unity with respect to $\{V_i\}$. For $u \in \mathscr{C}_0^\infty(K, E)$,

$$\|Pu\|_s = \sum_j \|h_j P u\|_s \le \sum_{i,j,k} \|h_j P h_k h_i u\|_s. \tag{1.3.78}$$

If $\operatorname{supp}(h_j) \cap \operatorname{supp}(h_i) \neq \emptyset$, $\operatorname{supp}(h_j) \subset U_i$. So $\|h_j P h_k h_i u\|_s \le \|h_i u\|_{s-m}$. If $\operatorname{supp}(h_j) \cap \operatorname{supp}(h_i) = \emptyset$, $h_j P h_k h_i$ is a smoothing operator. Thus

$$\|Pu\|_s \le \sum_{i,j,k} \|h_j P h_k h_i u\|_s \le C \sum_i \|h_i u\|_{s-m} = C\|u\|_{s-m}. \tag{1.3.79}$$

(2) follows from $\varphi Q P \psi = \sum_i \varphi Q h_i^{1/2} h_i^{1/2} P \psi$.

(3) follows from $\varphi P^* \psi = (\psi P \varphi)^*$.

(4) follows from $\phi^*[(\varphi \phi_* P \psi)u] = ((\phi^{-1})^* \varphi) P((\phi^{-1})^* \psi)(\phi^* u)$.

The proof of Theorem 1.3.19 is completed. □

In the rest of this section, we study the symbol of pseudodifferential operator on the manifold. We want to glue the symbols on each chart together.

We start from the differential operators. Let $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ be a differential operator such that on a coordinate chart U_i,

$$P|_{U_i} = \sum_{|\alpha| \le m} A_\alpha^{(i)} \frac{\partial}{\partial x^{\alpha,(i)}}. \tag{1.3.80}$$

What is the relation between $A_\alpha^{(i)}$ and $A_\beta^{(j)}$?

From (1.1.5), for example,

$$\begin{aligned} \frac{\partial}{\partial x_1^{(i)}} &= \frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \cdot \frac{\partial}{\partial x_{k_1}^{(j)}}, \\ \frac{\partial^2}{\partial x_2^{(i)} \partial x_1^{(i)}} &= \frac{\partial}{\partial x_2^{(i)}} \left(\frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \right) \cdot \frac{\partial}{\partial x_{k_1}^{(j)}} + \frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \frac{\partial x_{k_2}^{(j)}}{\partial x_2^{(i)}} \frac{\partial^2}{\partial x_{k_2}^{(j)} \partial x_{k_1}^{(j)}}, \cdots \end{aligned} \tag{1.3.81}$$

If the order is higher than one, terms are more and more crazy. We have to do the coordinate transformation modulo the $(m-1)$-th order terms. That is,

$$\frac{\partial^m}{\partial x^{(i)}_{l_1}\cdots\partial x^{(i)}_{l_m}} = \frac{\partial x^{(j)}_{k_1}}{\partial x^{(i)}_{l_1}}\cdots\frac{\partial x^{(j)}_{k_m}}{\partial x^{(i)}_{l_m}}\frac{\partial^q}{\partial x^{(j)}_{k_1}\cdots\partial x^{(j)}_{k_m}} + (m-1)\text{-order terms.} \tag{1.3.82}$$

Thus the only transformations of the top order are meaningful,

$$A^{(j)}_{k_1\cdots k_m} = A^{(i)}_{l_1\cdots l_m}\frac{\partial x^{(j)}_{k_1}}{\partial x^{(i)}_{l_1}}\cdots\frac{\partial x^{(j)}_{k_m}}{\partial x^{(i)}_{l_m}}. \tag{1.3.83}$$

As in (1.3.2), we write

$$\sigma_\xi(P)|_{U_j} = \sum_{|\alpha|=m} i^{|\alpha|}A^{(j)}_\alpha(x)\xi^{\alpha,(j)}. \tag{1.3.84}$$

If $\sigma_\xi(P)$ is globally defined, the necessary condition is

$$\xi^{l,(i)} = \frac{\partial x^{(j)}_k}{\partial x^{(i)}_l}\cdot\xi^{k,(j)} \tag{1.3.85}$$

From (1.1.14), we have

$$\xi^{l,(i)}dx^{(i)}_l = \xi^{k,(j)}dx^{(j)}_k. \tag{1.3.86}$$

Thus if $E = F = \mathbb{C}$, we can regard $\sigma_\xi(P)$ as a function on $\xi \in T^*M$. In fact, let $\{U_j \times \mathbb{C}^p\}$ be an atlas of T^*M with diffeomorphisms

$$\Phi_j : T^*M|_{U_j} \to U_j \times \mathbb{C}^p, \quad (x,\xi) \mapsto (\phi_j(x), \xi^{(j)}). \tag{1.3.87}$$

Then the transition function is $\mathrm{diag}\{\phi_{ij}, (D(\phi_{ij})^{-1})^T\}$. By (1.1.2), $\sigma_\xi(P)$ is a function on $\xi \in T^*M$. For smooth map $f : M \to N$, if $\pi' : E \to N$ is a vector bundle over N, we can define the pull-back bundle by

$$f^*(E) = \{(m,v) \in M \times E : f(m) = \pi'(v)\}. \tag{1.3.88}$$

Easy to see that f^*E is a vector bundle over M.

Let $\pi : T^*M \to M$ be the natural projection. In general, $\sigma_\xi(P) \in \mathscr{C}^\infty(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$, i.e., $\sigma_\xi(P)$ is a bundle map[2]

$$\sigma_\xi(P) : \pi^*E \to \pi^*F, \tag{1.3.89}$$

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[2] A bundle map is a map between two bundles over the same manifold which restricts to one point on the base manifold is a linear transform.

which is called the **principal symbol** of P. From (1.3.75), it is obvious that for differential operators $P, P' : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,F)$, $Q : \mathscr{C}^\infty(M,F) \to \mathscr{C}^\infty(M,G)$ and $t_1, t_2 \in \mathbb{C}$, we have

$$\begin{aligned} \sigma_\xi(t_1P + t_2P') &= t_1\sigma_\xi(P) + t_2\sigma_\xi(P'), \\ \sigma_\xi(Q \circ P) &= \sigma_\xi(Q) \circ \sigma_\xi(P). \end{aligned} \tag{1.3.90}$$

Now we extend the definition of the principal symbol to the pseudodifferential operator on manifold.

Let $P = \sum P_i \in \Psi\mathrm{DO}_m(E,F)$ and P_i is defined on U_i from the map $\phi_i : U_i \to \mathbb{R}^n$. From (1.3.36) and (1.3.76), after a diffeomorphism ϕ_{ij}, since $\chi(x,y) \equiv 1$ near the diagonal, we have

$$\mathrm{Sym}(\phi_{ij}^* P)(x^{(j)}, \xi^{(j)}) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha J(y) |\det\Theta| p(\phi_{ij}(x^{(j)}), \Theta\xi^{(j)})|_{y=x}, \tag{1.3.91}$$

where $\Theta = [(\partial x^{(i)}/\partial x^{(j)})^T]^{-1}$ and $J(y) = |\det(\partial x^{(i)}/\partial x^{(j)})_y|$. Then Since $|\det\Theta| = J(x)^{-1}$, we have

$$\mathrm{Sym}(\phi_{ij}^* P)(x^{(j)}, \xi^{(j)}) \sim p(x^{(i)}, \Theta\xi^{(j)}) \mod \mathrm{Sym}^{m-1}. \tag{1.3.92}$$

Therefore, as in (1.3.89), we could define the principle symbol $\sigma_\xi(P) \in \mathscr{C}^\infty(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$.

Definition 1.3.20. We say $p \in \mathscr{C}^\infty(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$ is a symbol of order m if p defines an element of Sym^m in each local coordinate chart. From Proposition 1.3.15, the definition is independent of the choice of the atlas. The vector space of all such symbols of order m is denoted by $\mathrm{Sym}^m(E,F)$.

If $P \in \Psi\mathrm{DO}_m(E,F)$, then $\sigma_\xi(P) \in \mathrm{Sym}^m(E,F)/\mathrm{Sym}^{m-1}(E,F)$.